A survey on Geometry of Interactions

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13 may 2010

References for G.o.I. (1)

- J.-Y. Girard. **Le point aveugle**, tome 1: vers la perfection, tome 2: vers limperfection. Visions des Sciences. Hermann, Paris, 2006-2007 (chapters 6, 9, 19).
- J.-Y. Girard: **Linear Logic, its syntax and semantics**. Advances in Linear Logic, eds Girard, Lafont, Regnier, London Mathematical Society Lecture Notes Series 222, Cambridge University Press 1995.
- J.-Y. Girard. **Towards a geometry of interaction**. In Categories in Computer Science and Logic, pages 69 108, Providence, 1989. American Mathematical Society. Proceedings of Symposia in Pure Mathematics n. 92.

References for G.o.I. (2)

- J.-Y. Girard. **Geometry of interaction I: interpretation of system F**. In Ferro, Bonotto, Valentini, and Zanardo, editors, Logic Colloquium 88, pages 221 260, Amsterdam, 1989. North-Holland.
- J.-Y. Girard. **Geometry of interaction II: deadlock-free algorithms**. In Martin-Löf and Mints, editors, Proceedings of COLOG 88, volume 417 of Lecture Notes in Computer Science, pages 76 93, Heidelberg, 1990. Springer-Verlag.
- J.-Y. Girard. **Geometry of interaction III: accommodating the additives**. In Girard, Lafont, and Regnier, editors, Advances in Linear Logic, pages 329 389, Cambridge, 1995. Cambridge University Press.

References for G.o.I. (3)

- J.-Y. Girard. **Geometry of interaction IV: the feedback equation**. In Stoltenberg-Hansen and Vnnen, editors, Logic Colloquium 03, pages 76 117. Association for Symbolic Logic, 2006.
- J.-Y. Girard. **Geometry of Interaction V: logic in the hyperfinite factor**. Fully revised version (October 2009).

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- It develops a logic of *actions*, i.e. non-reusable facts (vs. situations)
- The meaning of a formula is in its proof, not in its truth
- Each proof has and underlying **geometrical structure**
- G. of I. replaces static infinite situations by finite dynamic situations that lie in the geometrical structure of Gentzen's Hauptsatz

Basic logical commitments (1)

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- It is only through very heavy and *ad-hoc* paraphrases that real implication may be put into mathemtics (usually by adding a parameter for time)
- The logical laws extracted from mathematics are only adapted to eternal truth; the same principles applied in real life lead to absurdity because of the *interactive* nature of real implication

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- Geometry of interactions proposes: *proofs as actions*
- The logical twist from functions to actions leads to *linear logic*

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- To cope with situations, special connectives (*exponentials*) are needed: ! and ?, where ! means infinite iterability
- It is possible to define a new type of implication called *linear implication* ($\multimap$) such as:
  $A \Rightarrow B = (!A) \multimap B$

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- This corresponds to **IF** ... **THEN** ... **ELSE**

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- The meaning of $⅋$ is more complex and is related to *linear neagation*: $⅋$ is the symmetric form of linear implication

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- Example: **chemical equations**; in usual logic the phenomenon of *updating* cannot be represented

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- Example: **formal grammars**; it shows clearly the connection between linear logic and computation processes, that's why linear logic finds a natural application in computer science

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- It is more primitive (stronger) than usual negation

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- **Exchange**: it expresses the commutativity of multiplicative operators; it exists to achieve more expressive power

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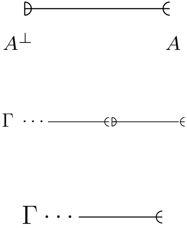
- Logical group will contain additives (disjunctions), multiplicatives (conjunctions) and exponentials (modalities)
- The only dynamical feature of the system is the cut-rule: it puts together an action and a reaction of the same type

Linear sequent calculus (3)

The calculus consist of replacing hypothesis with conclusions (*axioms*) and vice-versa (*cut-rule*):

An electrical representation of which is:



Proof-structures

If we accept two additional links:

$$\frac{A \qquad B}{A \otimes B}\ (\textit{times link}) \qquad\qquad \frac{A \qquad B}{A \parr B}\ (\textit{par link})$$

then we can associate any proof in a linear sequent calculus a graph-like proof structure.

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- Inside proof-structures let's call *proof-nets* those which can be obtained as the interpretation of sequent calculus proofs

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- This can be achieved by handling syntactic manipulations at the level of proof-nets
- This is done by means of the *cut-elimination* procedure, that has good properties:
  - it enjoys the Church-Rosser property (confluence);
  - it is linear in time;
  - the treatment of multiplicative fragment is purely local; in fact all cut-links can be simultaneously eliminated

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Coherent semantics

- The only existant semantics for mathematics are denotational (i.e. static), such as Scott semantics
- These semantics interpret proofs as functions instead of actions
- There is no name for such *operational semantics*, that's why Girard introduced the expression **GEOMETRY OF INTERACTIONS**
- The inadequation of the denotational semantics for computation becomes conspicuous if we note that such semantics will have a strong tendency to be infinite, whereas programs are finite dynamical processes

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- The problem with syntax is that it contains irrelevant informations of temporal nature
- It is therefore needed to find what's hidden behind syntax *without* going to denotation: a *non-subjectivistic approach to sense*
- There should be a purely *geometrical* notion of finite dynamical structure; in other words there should be a *geometrical* interpretation of Gentzen Hauptsatz

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- A *fact* is any subset of M equal to its orthogonal; it is immediate that $X \rightarrow Y$ is a fact is Y is a fact

Interpretation of the connectives

It's possible to interpret all the operations in linear logic by operations on facts:

1. **times** : $X \otimes Y := \{mn \,;\, m \in X \wedge n \in Y\}^{\perp\perp}$
2. **par** : $X ⅋ Y := (X^{\perp} \otimes Y^{\perp})^{\perp}$
3. **1** : $\mathbf{1} := \{1\}^{\perp\perp}$, where 1 is the neutral element of M
4. **plus** : $X \oplus Y := (X \cup Y)^{\perp\perp}$
5. **with** : $X \& Y := X \cap Y$
6. **zero** : $\mathbf{0} := \emptyset^{\perp\perp}$
7. **true** : $\top := M$
8. **of course** : $!X := (X \cap I)^{\perp\perp}$, where I is the set of idempotents of M which belong to **1**
9. **why not** : $?X := (X^{\perp} \cap I)^{\perp}$

It's easily seen that this semantics is sound and complete.

Coherent spaces

- A *coherent space* is a reflexive undirected graph. In other terms it consists of a set $|X|$ of atoms together with compatibility or *coherence* relation between atoms, noted $x \smile\frown y$ or $y \smile\frown x[modX]$ if there is any ambiguity as to X

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- Given a coherent space X, its linear negation $X\perp$ is defined by:
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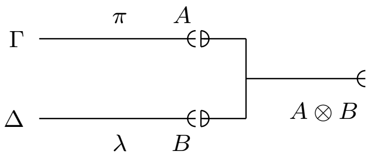
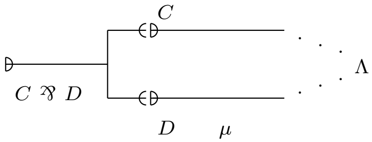
  - $x \smile\frown y[modX\perp]$ **iff** $x \smile\frown y[modX]$
- It is possible to define in this way all operators of linear logic (multiplicatives, additives, exponentials) to create a complete semantics

Geometry of interactions (1)

In a previous example we represented the links for proof-nets as electrical plugs; this is possible for all rules in linear logic, such as $\otimes - rule$ and $\invamp - rule$:





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- If we interpret the electrical encoders as $\otimes-$ or $\parr-$ link, we get a very precise modelization of cut-elimination of proof-nets
- Since this coding is based on the development by means of Fourier series (which involve Hilbert space), everything that was done can be formulated in terms of operator algebras

Geometry of interactions (3)

- The general formula for cut-elimination (*execution formula*) is the following:
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Geometry of interactions (3)

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- This gives the interpretation of the elimination of cuts (represented by ϕ) in a proof represented by u
- Termination of the process is interpreted as the nilpotency of ϕu and the part $u(1 - \phi u)^{-1}$ is a candidate for the execution

Interpretation of System F (1)

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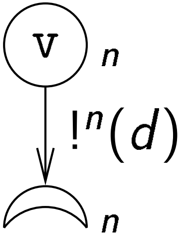
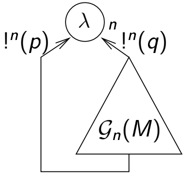
Interpretation of System F (2)

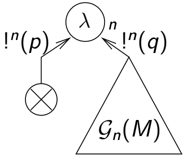
Variable case:



Interpretation of System F (3)

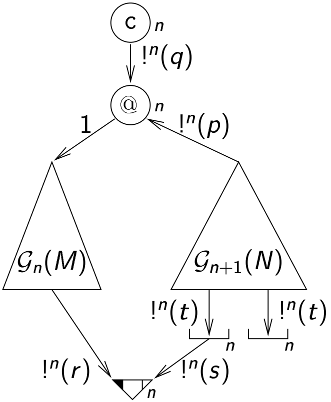
Abstraction:





Interpretation of System F (4)

Application:



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- Logical rules should not be symmetric w.r.t. time
- Processes communicates without understanding each other (by means of global operations such as erasing, duplicating, sending back)