ON THE GEOMETRIC INTERPRETATION OF SIGNALS

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This note will provide a brief introduction to the geometrical representation of signals. After some background material in linear algebra, the concepts of analysis and synthesis will be explained and the idea of convolution will be defined. More complete material on linear algebra and related to signals can be found in the references.

1. Vector spaces

Discrete signals x of length n can be thought as a multidimensional vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$. As such, they can be added with other vectors and multiplied by scalars as follows:

$$a_1 v_1 + a_2 v_2 \ldots + a_n v_n \tag{1}$$

where a_n are scalars and v_n are vectors.

Equation 1 is called a *linear combination* and a set of vectors that can be combined in such way form a *vector space*.

1.1. **Norm and metrics.**

On vector spaces can be computed several metrics:

- *mean*: $\mathcal{M} = \frac{1}{N}\sum_{n=0}^{N-1} x_n$;
- *energy*: $\mathcal{E} = \sum_{n=0}^{N-1} |x_n|^2$;
- *power*: $\mathcal{P} = \frac{\mathcal{E}}{N} = \frac{1}{N}\sum_{n=0}^{N-1} |x_n|^2$;
- L_2**-norm**: $\mathcal{N} = \sqrt{\mathcal{E}} = \sqrt{\sum_{n=0}^{N-1} |x_n|^2}$.

The latter metric, the norm, is often indicated as $\|x\|_2$ and represents the *length* of the vector in a space. When a vector space has a defined norm, it is called a *Banach space*. The L_2-norm is said to be *contractive*, that is:

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2 \tag{2}$$

and it can be generalized to any order: $L_p = \|x\|_p = \left(\sum_{n=0}^{N-1} |x_n|^p\right)^{\frac{1}{p}}$.

2. Inner product

A very important operation, called inner product, can defined on a Banach space as follows:

$$\langle x, y\rangle = \sum_{n=0}^{N-1} x_n \overline{y_n} \tag{3}$$

where $\overline{x}$ is the conjugate of x. A Banach space with a defined inner product is called a *Hilbert space*. The specific form of inner product shown in equation 3 is induced from the energy by taking the inner product of a vector with itself:

$$\langle x, x\rangle = \sum_{n=0}^{N-1} x_n \overline{x_n} = \sum_{n=0}^{N-1} |x_n|^2 = \|x\|_2^2. \tag{4}$$

2.1. **Properties.** The inner product has several important properties, among which:

- *Cauchy-Schwarz inequality*: $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$;
- *vector cosine*: $\frac{\langle x,y\rangle}{\|x\|\cdot\|y\|} \leq 1 = cos(\theta)$ where θ is the angle between the two vectors.

2.2. **Orthogonality.** Two vectors are said to be *orthogonal* (indicated as $x \perp y$) if their inner product is zero:

$$x \perp y \equiv \langle x, y\rangle = 0. \tag{5}$$

2.3. **Projection.** One of the most important applications of inner product is to project one vector over another. The projection of x on y is defined as:

$$\mathfrak{P}_x(y) = \frac{\langle x, y\rangle}{\|x\|^2} \cdot x \tag{6}$$

where the ratio between the inner product and the squared norm of x is called *coefficient of projection*. Important examples of projection will be shown in the following sections.

2.4. **Reconstruction from projections.** Under specific conditions, a vector can be reconstruced from some projections by means of linear combination.

Let $e_0(1,0)$ and $e_1(0,1)$ be to perpendicular vectors such as $\langle e_0, e_1\rangle = 0$ and let x be a distinct vector in $\mathbb{R}^2$. The projection of x on e_0 is given by:

$$\begin{aligned}\mathfrak{P}_{e_0}(x) &= \frac{\langle e_0, x\rangle}{\|e_0\|^2} \cdot e_0 = \langle e_0, x\rangle \cdot e_0 \\ &= (x_0 \cdot \overline{1} + x_1 \cdot \overline{0}) = (x_0, 0)\end{aligned}$$

and, in the same way, the projection on e_1 is given by $\mathfrak{P}_{e_1}(x) = (0, x_1)$.

It is indeed possible to *recover* x by summing the computed projections:

$$\begin{aligned}x = \mathfrak{P}_{e_0}(x) + \mathfrak{P}_{e_1}(x) &= x_0 \cdot e_0 + x_1 \cdot e_1 \\ &= x_0 \cdot (1,0) + x_1 \cdot (0,1) = (x_0, x_1).\end{aligned}$$

It is important to remark that this reconstruction only works if the vectors on which the projections are done are pairwise orthogonal; in case the used vectors for projections are linearly independent but not orthogonal, they can be orthogonalized by a method called *Gram-Schmidt orthogonalization.* Finally, if $\|e_n\|^2 = 1$ the set of vectors are called *orthonormal.*

2.5. **Basis.** The subspace covered by all linear combinations of a set of vectors $\{s_0, \ldots, s_n\}$ is called *span*. If the set of vectors are linearly independent than the span is called *basis* of the vector space. It is easy to show that in a space in $\mathbb{R}^d$ there are d vectors in the basis for that space. Clearly, a vector can be reconstructed with a linear combination from its projections on another set of vectors if and only if the set used is a basis.

3. Analysis and synthesis

Previous sections showed that a vector in vector space can be written as a linear combination of a basis for that space, by multiplying it by some constants and summing the products.

It is therefore possible to define an **analysis** as the estimation of the constants and a **synthesis** as the linear combination equation that recover the signal; following subsections will clarify this important concept.

3.1. **Analysis.** The analysis is the representation ϕ_x of a signal given by the inner product of it by a basis in a vector space; it is therefore given by the projection

$$\phi_x = \sum_t x(t) * \overline{b_k} = \langle x, b_k \rangle \tag{7}$$

where b_k is a given basis and t is time.

3.2. **Synthesis.** The synthesis is the reconstruction of the original signal x by the summation of the products with the representation ϕ_x created by the analysis:

$$x(t) = \sum_k \phi_x b_k(t) = \sum_k \langle x, b_k \rangle b_k(t). \tag{8}$$

3.3. **The discrete Fourier transform.** The Fourier representation is interpretable in the following context as a specific case of analysis and synthesis, where the basis is given by a set of complex sinusoids: $b_k = e^i 2\pi k$ (where i is the imaginary unit).

The discrete Fourier analysis (DFT) will be therefore:

$$\boxed{\hat{x}(k) = \sum_t x(t) e^{\frac{-i2\pi kt}{T}}} \tag{9}$$

and, in the same way, the reconstruction (or inverse Fourier transform, IDFT) is given by:

$$\boxed{x(t) = \frac{1}{T} \sum_k \hat{x}(k) e^{\frac{i2\pi kt}{T}}}. \tag{10}$$

Since $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ by Euler identity, the basis given by complex sinusoids made of linearly independent vectors); for this reason it is possible to recover exactly the original signal as explained in section 2.4. Note, moreover, that the negative i is forward Fourier analysis represents the conjugate of the basis, as prescribed by equation 3. The normalization factor $\frac{1}{T}$ is necessary in order to preserve the contraction property indicated in section 1.1.

3.4. Different bases.

The basis made of complex sinusoids is only a possible basis among infinite. This basis focus on representig correctly frequencies and is therefore well localized in frequency but is not localized in time. On the other hand, it is possible to create a basis made of Dirac's pulses that will provide perfect localization in time but no localization in frequency.

A compromise between sinusoids and impulses is given by bases made of oscillating signals with a temporal limitation, such as *wavelets*. A wavelet is a bandpass filter centered on a specific frequency with a specific bandwidth that has therefore a localization both in time and frequency. The Gabor wavelet represent the best compromise, in term of Heisenberg uncertainty principle, between time and frequency. More information on this vast subject can be found in the references.

4. Convolution

Convolution is a mathematical operation defined in vector spaces that has important applications in signals theory and it can be defined in term of inner product. In general it is possible to say that the inner product between two vectors is:

1. the projection of a vector onto the other as discussed in previous sections (or, in other words, the product of magnitudes scaled by the angle between the vectors);
2. the sum of the elements of a vector, weighted by the elements of the other;
3. the calculation of the *similarity* (covariance) of the two vectors.

The convolution then, can be defined as an inner product that is repeated over time:

$$(x * h)_t = \sum_m x(t - m) * h(m) \tag{11}$$

where h is called *kernel* and is of length m. The convolution is therefore:

1. the time series given by a signal weighted by another that slides along;
2. the cross-variance between two signals (similarity in time);
3. the time series given the mapping between to signals;
4. a filtering process.

4.1. The convolution theorem.

There is an important relation between the DFT and convolution: the convolution in time domain between to signals is equal to the product of the DFT of them. Formally:

$$x * h \equiv \hat{x} \cdot \hat{h} \tag{12}$$

where $\hat{x}, \hat{h}$ are DFTs of respective signals.

References

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[3] C. E. Cella, Chi ha paura dei numeri complessi?, unpublished.
[4] S. Mallat, A wavelet tour of signal processing, third edition.