# Building representations by factorization

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## 1 Introduction

Let $S = \{x_n\}_{n \leq N}$ be a set of obseravations of in $\mathbb{C}^d$. A *convolutional representation* $\Phi x_n$ of an element of $S$ is given by:

$$\Phi x_n = \{x_n * g_k\}_{k \leq K, n} \tag{1}$$

where $G = \{g_k\}_{k \leq K}$ are the bases of the representation. In the case of the Fourier representation, for example, equation 1 becomes:

$$\hat{X} x_n = \{x_n * e^{-2\pi jk}\}_{k \leq K, n} \tag{2}$$

where $K$ is a set of frequencies. The set $G$ can be given beforehand (as in the above case) or can be learnt on a set of signals by creating a global mapping $\mathcal{M}$ such as:

$$\{x_n\}_{n \leq N} \xrightarrow{\mathcal{M}} \{g_k\}_{k \leq K}. \tag{3}$$

Discovering such mapping can be very complex and often deals with problems in high dimensions.

## 2 Factorization

A possible way to build the mapping of equation 3 relies on reducing the dimensionality of the problem by factoring $\{x_n\}_{n \leq N}$. Let $L'$ be a *local mapping* such as:

$$\{x_n\}_{n \leq N} \xrightarrow{L'} \{f_j^n\}_{j \leq J^n, n} \tag{4}$$

where $F^n = \{f_j^n\}_{j \leq J^n, n}$ is a set of *signal-dependent* bases that are built by independently studying any element of $\{x_n\}_{n \leq N}$. Since looking at a single $x_n$ simplifies dramatically the problem, finding $L'$ is not usually difficult. Morerover, since $F^n \subset S$, its dimensionality is $\mathbb{C}^{d/q}$ where $q$ will depend on the cardinality of $F^n$. While it is also possible to create a signal-dependent representaion $\Phi^n x_n$ as follows:

$$\Phi^n x_n = \{x_n * f_j^n\}_{j \leq J^n, n} \tag{5}$$

this kind of representation cannot be used for signals other than $x_n$ and is not very useful. For this reason, it is necessary to create a global mapping $M'$ (that must not be signal-dependent) such as:

$$\{f_j^n\}_{j\leq J, n\leq N} \xrightarrow{M'} \{g_k\}_{k\leq K}. \tag{6}$$

Because of the reduced dimensionality of the problem created by factorization, finding the global mapping $M'$ could resolve in being an easier problem than finding $\mathcal{M}$ directly. Since it holds that $\{L' \circ M'\} \subseteq \mathcal{M}$, the bases found by factorization can be as optimal as the bases found by working directly on $\{x_n\}_{n\leq N}$.

This process can be iterated several times, in order to approximate the original dimensionality of the problem with layers of local and global mappings (where global mappings can be regarded as *pooling* operators):

$$\{x_n\}_{n\leq N} \xrightarrow{L'} \{f_j^n\}_{j\leq J, n\leq N} \xrightarrow{M'} \ldots \xrightarrow{M^h} \{g_k\}_{k\leq K}. \tag{7}$$

The complete representation, then, will be given by the following composition:

$$\{L' \circ M' \circ \ldots \circ M^h\} \subseteq \mathcal{M}. \tag{8}$$

where the last element of the composition must be a global mapping.

# 3 Criteria

The difficult part of this process is to find good criteria for the factorizations at each layer. One possible way is to use *ergodicity* as a general guideline for local mappings and geometry for global mappings: each local mapping should group together things that have the same ergodicity, while each global mapping should project all the bases on simpler spaces.